Study of F-threshold of Rings of Characteristic p

Dissertation submitted to
Visvesvaraya National Institute of Technology, Nagpur
in partial fulfillment of the requirement for the award of
the degree

Master of Science
in
Mathematics

by

BASANT KUMAR
(MS23MTH023)

under the guidance of

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Visvesvaraya National Institute of Technology
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Declaration

I, Basant Kumar, hereby declare that this dissertation titled **"Study of F-threshold of Rings of Characteristic p"** is carried out by me at the Department of Mathematics of Visvesvaraya National Institute of Technology, Nagpur. The work is original and has not been submitted in whole or in part to this or institutions/universities for the awarding of a degree/diploma.

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This is to certify that the dissertation entitled "Study of F-threshold of Rings of Characteristic p" is a bonafide work done by **Basant Kumar** (Enrollment No. MS23MTH023). The project work has been completed under my guidance and supervision satisfactorily. This work is a part of the requirement for the award of **Master of Science in Mathematics** from the Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur. This dissertation fulfills the requirements relating to the nature and standard of work for the award of a Master of Science degree.

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Basant Kumar

Abstract

This dissertation explores the study of F-threshold of rings of characteristic p. We start with some important preliminaries and definitions required to better understanding of the F-threshold and F-pure threshold. Further, we discuss about the relation between F- threshold and F- pure threshold. In the subsequent phase, we discuss about the computation of F-threshold of determinantal rings with matrices for 3×3 case. After that using this approach, we subsequently extended for 4×4 matrices.

Notations

$S = K[Y]$	Polynomial ring over a field K with indeterminates y_{ij}
$I_l(X)$	Ideal generated by all $l \times l$ minors of Y
$R = S/I_l(Y)$	Determinantal ring defined by the ideal $I_l(Y)$
$\mathfrak{m}$	Homogeneous maximal ideal of S or R
$\mathfrak{m}^{[q]}$	Frobenius power of $\mathfrak{m}$, i.e., generated by the q-th powers of the generators
$c^J(a)$	F-threshold of ideal a with respect to J
$c^{\mathfrak{m}}(\mathfrak{m})$	F-threshold of the maximal ideal $\mathfrak{m}$
$a(R)$	The a-invariant of the graded ring R
$\mathrm{fpt}(f)$	F-pure threshold of a polynomial f
$M(g)$	Exponent matrix of a monomial g
$\in$	Belongs to
$\forall$	For all
$\Rightarrow$	Implies to
$\mapsto$	Maps to
$\subset$	Subset
$\sim$	Equivalence

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Chapter 1

Introduction

Commutative algebra is the study of commutative rings and their properties. The main figure responsible for the birth of commutative algebra as a mature subject was Wolfgang Krull($1920-1940$), who introduced the fundamental notions of localization and completion of a ring, as well as that of regular local rings. Initially known as ideal theory, it began with the "Study on Richard Dedekind's Ideals" and is based on early work by Ernst Kummer and Leopold Kronecker [8]. David Hilbert further developed the concept by generalizing the notion of rings beyond numbers. Commutative algebra involves key concepts such as localization, Noetherian and Artinian rings, and absolute values, and it plays a crucial role in the local study of schemes. Important examples of commutative rings include the set of integers and polynomial rings of algebraic integers. The central concept in commutative algebra is that of prime ideals. This field is foundational for studying other areas of mathematics, particularly in understanding the geometric properties of manifolds and singular spaces. In contemporary times, commutative algebra is extensively used in computational aspects of both engineering and science. For instance, combinatorial objects like graphs and Gröbner bases are applied in various engineering calculations, including string theory. In recent decades, computational advances have significantly impacted commutative algebra, with applications in cryptography and coding theory. The field continues to explore connections with algebraic geometry, number theory, and homological algebra, demonstrating its enduring relevance and dynamic evolution. From ancient mathematical techniques to modern computational applications, the history of commutative algebra is a testament to its foundational role in mathematics.
Commutative algebra and algebraic geometry have seen a lot of research in recent years, which has helped to clarify prime characteristic singularities in local theory. These days, singularities are called F-singularities, which are defined in terms of the

behavior of the Frobenius endomorphism. Special attention to the four most popular F-singularity classes: F-pure, F-injective, strongly F-regular, and F-rational singularities. How good or bad is a singular point based on the most singular singularity that can exist? Important numerical invariants of positive characteristic rings that quantify the behavior of these singularities are the F-threshold, F-pure thresholds, and F-signature. Mustaţa, Takagi and Watanabe initially proposed the F-threshold for regular local ring [5]. Huneke, Mustaţa, Takagi and Watanabe expand this invariant to a more general and related with tight closure, integral closure and multiplicities [9]. Moreover, Chiba and Matsuda calculated the F-threshold of Hibi rings which are a type of graded toric rings on a finite distributive lattice. F-threshold is a numerical measure that helps understand how certain algebraic structures behave, especially in rings with positive characteristic. In general it is difficult to find F-threshold of large classes of rings. In this dissertation, we learn the relation between F-threshold and F-pure threshold. We also discuss computation of F-threshold of a special class of ring, called determinantal ring.This dissertation explores the study of F-threshold of rings of characteristic p. In Chapter 2, we recall the basic definitions and fundamental results of Commutative algebra. In Chapter 3, we discuss about the relation between F-threshold and F-pure threshold. In Chapter 4, we study about the computation of F-threshold of Determinantal rings for special cases.

Chapter 2

Preliminaries

This chapter revisits basic definitions and fundamental results of commutative algebra. Subsequently, it delves into concepts such as Graded rings and Module. Definitions and results in this context are primarily derived from book [1].

Definition 2.1. ***(Commutative Ring)*** *Let S be a non empty set equipped with two operations one denoted by addition ('+') and another denoted by multiplication ('.') such that*
1. *S is forms an abelian group with respect to addition.*
2. *'.' is associative i.e, $(x.y).z = x.(y.z)$ for all $x, y, z \in S$*
3. *'+' is distributive i.e, i. $(x + y).z = (x.y) + (y.z)$*
ii. $x.(y + z) = (x.y) + (y.z)$ for all $x, y, z \in S$
4. *'.' is commutative i.e, $x.y = y.z$ for all $x, y \in S$*
5. *There exists 1 in S such that $x.1 = 1.x = x$ for all $a \in S$ then S is called commutative ring with unity.*

Example 2.1. *(1). $\mathbb{Z}/6\mathbb{Z}$ is a commutative ring.*

Definition 2.2. ***(Subring)*** *A subring is a subset of ring S that is also form a ring for the same operation which is defined on S.*

Example 2.2. *$4\mathbb{Z}$ is subring of $\mathbb{Z}$.*

Definition 2.3. ***(Ideal)*** *Given S be a ring. A subset $I \subseteq S$ is a **left ideal** if it satisfies the following two conditions:*

1. *$(I, +)$ is a subgroup of $(S, +)$.*
2. *For all $r \in S$ and $x \in I$, the product $r \cdot x \in I$. (Left absorbs)*

3. *For all $r \in S$ and $x \in I$, the product $x \cdot r \in I$ (Right absorbs). If left ideal and right ideal are equal, then I is called ideal of ring S.*

Example 2.3. *Let $\mathbb{Z}[\mathbb{X}]$ be a ring and $I =< x, 5 >$ be an ideal of $\mathbb{Z}[\mathbb{X}]$.*

Definition 2.4. ***(Quotient Ring)*** *Given any ring(take S be a ring) and let I be any ideal of S. The set of cosets $\{x + I | x \in S\}$ form a ring under the operations $(y + I) + (z + I) = y + z + I$ and $(y + I) \cdot (z + I) = y \cdot z + I$, then S/I is called Quotient Ring.*

Example 2.4. *Let $R[y]$ be the ring of polynomials with coefficients from $\mathbb{R}$. Define the ideal:*

$$I = \langle y^2 + 1 \rangle = \left\{ (y^2 + 1)q(y) \,\middle|\, q(y) \in R[y] \right\}.$$

Then quotient ring of $R[y]$ is given by:

$$R[y]/\langle y^2 + 1 \rangle.$$

Definition 2.5. ***(Ring Homomorphism)*** *Let R and S be two rings, then ring homomorphism is a operation preserves for addition, multiplication and multiplicative identity; like this,*
Let $\varphi : R \to S$ be any map.

$$\begin{aligned} \varphi(a + b) &= \varphi(a) + \varphi(b), \\ \varphi(ab) &= \varphi(a)\varphi(b), \\ \varphi(1_R) &= 1_S, \end{aligned}$$

for all $a, b \in R$.

Example 2.5. *The function $f : \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z}$, defined by*

$$f(a) = a \bmod 5,$$

is a ring homomorphism.

Definition 2.6. ***(Maximal ideal)*** *Let S be a ring, and let I be a proper ideal in S such that $I \neq S$. We say that I is a* maximal ideal *of S if and only if the following equivalent conditions are satisfied:*

1. *There is no ideal J such that $I \subsetneq J \subsetneq S$. In other words, there is no ideal that lies strictly between I and S.*

2. *If J is any ideal of S such that $I \subseteq J$, then J must be either I or S.*

Example 2.6. *Ideal generated by* $(11, x)$ *is a maximal ideal in the polynomial ring* $\mathbb{Z}[x]$.

Definition 2.7. ***(Monomial)*** *A monomail in* $z_1, z_2, \ldots, z_n$ *is a product of the form* ${z_1}^{\beta_1} \cdots {z_n}^{\beta_n}$, *where* $\beta_1, \cdots, \beta_n$ *are positive integers.*

Example 2.7. $z_1 z_2$, ${z_1}^3 z_2 z_3$

Definition 2.8. ***(Radical of an ideal)*** *Let* S *be a commutative ring and take* $I \subseteq S$ *be an ideal. The* ***radical*** *of an ideal* I, *denoted by* $\sqrt{I}$, *is defined as:*

$$\sqrt{I} = \left\{ s \in S \mid s^n \in I \text{ for some } n \in \mathbb{Z}^+ \right\}.$$

In words, the radical of I *consists of all elements* $s \in S$ *such that some positive power of* s *belongs to* I.

Example 2.8. *The radical of* $28\mathbb{Z}$ *is* $14\mathbb{Z}$.

Definition 2.9. ***(Characteristic of ring)*** *The smallest positive number* n *such that* $n * 1 = 0$ *is a characteristic of a ring* R, *where* 0 *is the additive identity and* 1 *is the multiplicative identity.*
if there is number n, *otherwise* 0.

Example 2.9. Z_5 *has characteristic is* 5.

Definition 2.10. ***(Perfect field)*** *A field* F *is called perfect if either of the following conditions holds:*

1. *The field has characteristic 0.*
2. *The field has characteristic* p *(a prime number) and every element in* F *is* x p*-th power, meaning for every* $x \in F$, *there exists some* $y \in F$ *such that* $x = y^p$.

In other words, a field is perfect if it contains no inseparable extensions.

Example 2.10. *1. Set of rational number is field with characteristic zero.*

Definition 2.11. ***(*R***-Module)*** *Let* R *be Commutative ring with unity. A set* M *with a Binary Operation '+' is said to be a* R*-module if* $(M,+)$ *is an abelian group and there exists a map* $R \times M \to M$ *such that* $(a,m) \longmapsto a \cdot m$ *and following these properties:*

1. $a \cdot (m_1 + m_2) = a \cdot m_1 + a \cdot m_2$

2. $(a+b)\cdot m = a\cdot m + b\cdot m$

3. $(a\cdot b)\cdot m = a\cdot(b\cdot m)$

4. $1\cdot m = m \;\forall\; m, m_1, m_2 \in M$ *and for all* $a, b \in R$.

Example 2.11. $\mathbb{Z}_5$ *is module over itself.*

Definition 2.12. ***(R-Module Homomorphism)*** *Let* $\varphi : M \to N$ *be a function between two modules over a ring* R. *We say that* φ *is an* R***-module homomorphism*** *(or an* R***-linear map***) *if it satisfies the following two properties for all* $x, y \in M$ *and* $r \in R$:

$$\varphi(x+y) = \varphi(x) + \varphi(y),$$
$$\varphi(r\cdot x) = r\cdot\varphi(x).$$

Example 2.12. *Consider the function:*

$$\varphi : R \to R, \quad x \mapsto cx$$

φ *is* R*-module homomorphism*

A-Algebra

Let $\varphi : A \to B$ be a ring homomorphism. If $a \in A$ and $b \in B$, we define scalar multiplication as:

$$a\cdot b = \varphi(a)b$$

This definition provide the ring B with the structure of an A-module. Since B already has a ring structure, and now an A-module structure that's compatible with it, we say that B is an **A-algebra**.

Definition 2.13. *Let* $\varphi : A \to B$ *be a ring homomorphism. The homomorphism* φ *is said to be* ***finite****, and the ring* B *is called a* ***finite*** A***-algebra****, if* B *is finitely generated as an* A*-module.*

Moreover, B is a **finitely generated A-algebra** if there exist elements $x_1, \ldots, x_n \in B$ such that every element in B can be written as a polynomial in $x_1, \ldots, x_n$ with coefficients in $\varphi(A)$.

Definition 2.14. ***(Tensor Product of Modules)*** *Let R be a commutative ring and let M and N be R-modules. The* ***tensor product*** *of M and N, denoted $M \otimes_R N$, is an R-module along with a bilinear map*

$$\tau : M \times N \to M \otimes_R N$$

that satisfies the following ***universal property****:*

Given any R-module P, and any R-bilinear map $f : M \times N \to P$, there exists a unique R-linear map

$$\tilde{f} : M \otimes_R N \to P$$

such that the composition satisfies:

$$f = \tilde{f} \circ \tau.$$

Definition 2.15. ***(Noetherian Ring)*** *A ring R is said to be* ***Noetherian*** *if it satisfies the following three equivalent conditions:*

1. *Every non-empty set of ideals in A has a maximal element.*
2. *Every ascending chain of ideals in A is stationary.*
3. *Every ideal in A is finitely generated.*

Example 2.13. *The ring $\mathbb{Z}/5\mathbb{Z}$ is* ***Noetherian****.*

Example 2.14. *Every field $\mathbb{F}$ is Noetherian ring as it has only two ideals (0) and itself.*

Proposition 1. *[2] A module M over a ring A is Noetherian if and only if every submodule of M is finitely generated.*

Proof. Suppose M is Noetherian, then every submodule of M is finitely generated. Let $N \subseteq M$ be any submodule. Define the set

$$\Sigma = \{N' \subseteq N \mid N' \text{ is finitely generated}\}.$$

Clearly, $0 \in \Sigma$, so Σ is non-empty. Since M is Noetherian, we can apply Zorn's Lemma to conclude that Σ contains a maximal element, say N_0.

Assume $N_0 \neq N$. Then there exists some element $x \in N \setminus N_0$, and the submodule $N_0 + Rx$ strictly contains N_0. Since N_0 is finitely generated and Rx is generated by

one element, their sum is also finitely generated. This contradicts the maximality of N_0, hence $N = N_0$. So N is finitely generated. Suppose every submodule of M is finitely generated. Then M is Noetherian. Consider any ascending chain of submodules $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$.

Let $N = \bigcup_{i=1}^{\infty} M_i$. By assumption, N is a submodule of M and must be finitely generated, say by $\{x_1, \ldots, x_k\}$. Each x_j lies in some M_{n_j}, so all generators of N are contained in M_N for some large enough N. This implies $M_n = N$ for all $n \geq N$, so the chain stabilizes. Therefore, M satisfies the ascending chain condition, and hence M is Noetherian.

□

Proposition 2. *[2] Let A be a noetherian ring, M a finitely generated A-module. then M is noetherian.*

Theorem 2.1. ***(Hilbert's Basis Theorem)*** *[2] If A is noetherian, then the polynomail ring ring $A[x]$ is noetherian.*

Proof. Proof by contradiction, Suppose $\exists$ an ideal $I \subset A[x]$ which is not finitely genereted set $I_0 = (0)$. Let $f_1 \in I$ be the polynomial in I of least degree. Let $I_1 = (f_1)$ let f_2 be an element of least degree in $I/(f_1)$ and $I_2 = (f_1, f_2)$ recursevely. let f_i be a polynomial of least degree in $I/(f_1.....f_{i-1})$ then observe that,
(1) $degf_1 \leq degf_2 \leq degf_3 \leq .....$
(2) $I_1 \subseteq I_2 \subseteq I_3 \subseteq .....$
Let a_i be the leading coefficient of f_i and let J_i= $(a_1.....a_i)$ such that $J_1 \subseteq J_2 \subseteq J_3 \subseteq$ .....
Since this is chain of ideal in A, Then $\exists$ n such that $J_n = J_{n+1} =$ .....
which means that: $a_{n+1} = \sum_{i=1}^{n} r_i a_i$; for some $r_i \in A$
Let $f = f_{n+1} - \sum_{i=1}^{n} r_i f_i$, then $degf \leq degf_{n+1}$
$f \in I_{n+1}$ which implies that $f \in I_n$ (since f_{n+1} is the polynomial of least degree $I/(f_1.....f_n)$
then, we get $f_{n+1} \in I_n$, which is contradiction.
So $A[x]$ is noetherian ring.

□

Definition 2.16. ***(Graded ring)*** *A graded ring is a ring A together with a family $(A_n)_{n\geq 0}$ of subgroups of the additive group of A, such that $A = \oplus_{n=0}^{\infty}(A_n)$ and $A_m A_n \subseteq A_{m+n}$ for all m, $n \geq O$. Thus A_0 is a subring of A, and each A_n is an A_o-module.*

Example 2.15. *$A = k[x_1 \cdots x_r]$, A_n= set of all homogeneous polynomials of degree n. Then, $A = \oplus_{n=0}^{\infty}(A_n)$ and $A_m A_n \subseteq A_{m+n}$ for all $m, n \geq 0$ so that, $(A_n)_{n\geq 0}$ is a gradation.*

Proposition 3. *Let $A = \bigoplus_{g\in G} A_g$ be a graded ring by an abelian group G. Then A_0 is a ring with identity. In fact, $1_A \in A_0$ and each A_g is an A_0-module.*

Proof. We will prove that $1_R \in R_0$. As $R = \bigoplus_{i=1}^{n} R_{g_i}$, we can write:

$$1_R = e_{g_1} + e_{g_2} + \cdots + e_{g_n}$$

where $e_{g_i} \in R_{g_i}$ and $e_{g_i} \neq 0$.

Now, multiplying both sides by e_{g_i}, we get:

$$e_{g_i} = e_{g_i}(e_{g_1} + e_{g_2} + \cdots + e_{g_n}) = \sum_{j=1}^{n} e_{g_i} e_{g_j}$$

By the properties of the direct sum decomposition, the product $e_{g_i} e_{g_j} \in R_{g_i+g_j}$, and since $e_{g_i} \in R_{g_i}$, the only non-zero term in the sum must lie in R_{g_i}. Thus:

$$e_{g_i} = e_{g_i} e_{g_j} \text{ for some } j, \quad \text{and} \quad e_{g_i} e_{g_k} = 0 \text{ for all } k \neq j$$

Note that $g_i + g_j = g_i$, so $g_j = 0$. Without loss of generality, let $j = 1$. Therefore, we conclude:

$$1_R = e_0 + e_{g_2} + \cdots + e_{g_n}$$

Now consider:

$$1 = 1^2 = (e_0 + e_{g_2} + \cdots + e_{g_n})^2 = \sum_{i,j} e_{g_i} e_{g_j}$$

But again, by the grading properties:

$$e_{g_i} e_{g_j} \in R_{g_i+g_j}$$

Thus, the only way this sum lies in R_0 is if all products $e_{g_i} e_{g_j}$ are zero unless $g_i + g_j = 0$. In particular, $e_0^2 = e_0 \in R_0$, and all other terms must vanish or lie in different components. Hence:

$$1 = e_0 \in R_0$$

□

Definition 2.17. ***(Graded R-Module)*** *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring, and let M be an R-module. We say that M is a* ***graded R-module*** *if there exist abelian subgroups M_g of M for all $g \in G$ such that:*

$$M = \bigoplus_{g\in G} M_g \quad \textit{and} \quad R_h M_g \subseteq M_{h+g} \quad \textit{for all } h, g \in G.$$

Example 2.16. *If $\{M_i\}_{i\in I}$ is a family of graded R-modules, then both the direct sum $\bigoplus_{i\in I} M_i$ and the product $\prod_{i\in I} M_i$ are graded R-modules in the obevious way, then R^n is a graded R-module.*

Definition 2.18. ***(Dimension of Ring)*** *Let R be a ring. The* dimension *of R, denoted by* $\dim R$*, is the supremum of the lengths n of strictly ascending chains of prime ideals of the form:*

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n,$$

where each $\mathfrak{p}_i$ is a prime ideal of R, and $\subsetneq$ denotes proper inclusion.

$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ *in R. That is,*

$$\dim R = \sup\left\{n \in \mathbb{N} \,|\, \exists\ \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n\right\}.$$

Example 2.17. $\mathbb{Q}$ *is field with dimension zero.*

Chapter 3

F-threshold and F-pure threshold

In the study of singularities in rings of prime characteristic $p > 0$, F-threshold and F-pure threshold are numerically invariants measure the severity of singularities. In this chaper, we study of a connection between two invariants F-threshold and F-pure threshold with characteristic greater than zero.

Definition 3.1. ***(Frobenius map)*** *Let S be a commutative ring of prime characteristics p. The p^{th} power map is the Frobenius map:*

$$F : S \to S,\ r \mapsto r^p.$$

The Frobenius map is a ring homomorphism because $(r+s)^p = r^p + s^p$ in characteristic p. The subring S^p of all S elements that are p^{th} powers is its image. As a result, the rings $S^p \mapsto S$ are included

Example 3.1. *Consider a polynomial $f(x) \in S$. Let us write $f(x)$ explicitly as: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where $a_i \in F_p$ for all i. Applying the Frobenius map to $f(x)$ gives:*

$$F(f(x)) = f(x)^p = (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)^p$$

since the coefficients a_i are in the field F_p, we have $a_i^p = a_i$ because F_p is a perfect field. Using the binomial theorem and the fact that binomial coefficients $\binom{p}{k}$ are divisible by p for $0 < k < p$, we get $(a_ix_i)^p = a_i^p x^{ip} = a_ix^{ip}$.
Therefore the polynomial $f(x)^p$ can be written as: $F(f(x)) = f(x)^p = a_0^p + (a_1x)^p + (a_2x^2)^p + \cdots + (a_nx^n)^p = a_0 + a_1x^p + a_2x^{2p} + \cdots + a_nx^{np}$.

Lemma 1. *Let S be a commutative ring of characteristic $p > 0$. Then Frobenius map $F : S \to S, \quad a \mapsto a^p$ is a ring homomorphism.*

Proof. Let $a, b \in S$. To show that F is a ring homomorphism

$$F(a+b) = F(a) + F(b)$$

By the binomial theorem, we have:

$$(a+b)^p = \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i}$$

Since the characteristic of S is p, it follows that $p \cdot 1_S = 0$, and hence the binomial coefficients satisfy:

$$\binom{p}{i} \equiv 0 \pmod{p} \quad \text{for } 1 \leq i \leq p-1$$

Therefore, all intermediate terms vanish, and we obtain:

$$(a+b)^p = a^p + b^p$$

Hence,

$$F(a+b) = F(a) + F(b)$$

also

$$F(ab) = (ab)^p = a^p b^p = F(a)F(b)$$

This follows directly from the fact that exponentiation distributes over multiplication in any commutative ring. For multiplicative identity

$$F(1_S) = (1_S)^p = 1_S$$

Since F respects addition, multiplication, and the multiplicative identity, it follows that F is a ring homomorphism □

Example 3.2. *Let $S = \mathbb{F}_3[x]$ be the ring of polynomials over the finite field $\mathbb{F}_3 = \{0, 1, 2\}$, which has characteristic $p = 3$. Define the Frobenius map by*

$$F : S \to S, \quad f(x) \mapsto f(x)^3.$$

Let $f(x) = x + 1$, and $g(x) = x^2 + 2$. Then F is a ring homomorphism.

Lemma 2. *Let S be a ring of prime characteristic $p > 0$. Then S is reduced if and only if $F : S \to S$ is injective.*

Proof. Suppose S is reduced, i.e., S has no nonzero nilpotent elements. Let $r \in S$ be such that $F(r) = r^p = 0$. Then r is nilpotent, because in characteristic p, $r^p = 0$ implies r is nilpotent. Since S is reduced, it follows that $r = 0$. Hence, F is injective.

Conversely, suppose F is injective. Let $r \in S$ be nilpotent, say $r^n = 0$ for some $n > 0$. Choose $q = p^e$ for some $e \in \mathbb{N}$ such that $q \geq n$. Then $r^q = 0$, so $F^e(r) = r^q = 0$. Since F is injective, all its iterates are injective, so $r = 0$. Therefore, S has no nonzero nilpotent elements, and is reduced. □

Definition 3.2. ***(Determinantal ring)****Let K be an F -finite field of characteristic $p > 0$, and let $Y = (y_{ij})$ be an $p \times q$ matrix of indeterminates. Consider the polynomial ring $S = K[Y]$ over the entries of Y . Given $l \leq p \leq q$, we take the ideal I_l of the $l \times l$ minors of Y and the corresponding determinantal ring $R = S/I_l$.*

Example 3.3.

- *Let R be the polynomial ring over a field k:*

$$R = k[y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}]$$

Consider the 2×3 matrix:

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix}$$

where the y_{ij} are indeterminates.

- *A minor is the determinant of a square submatrix. Since Y has two rows, we can form three 2×2 minors by selecting pairs of columns:*

Minor from the first two columns:

$$\det \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = y_{11}y_{22} - y_{12}y_{21}$$

Minor from the first and third columns:

$$\det \begin{bmatrix} y_{11} & y_{13} \\ y_{21} & y_{23} \end{bmatrix} = y_{11}y_{23} - y_{13}y_{21}$$

Minor from the second and third columns:

$$\det \begin{bmatrix} y_{12} & y_{13} \\ y_{22} & y_{23} \end{bmatrix} = y_{12}y_{23} - y_{13}y_{22}$$

The ideal I generated by these minors is:

$$I_l =< y_{11}y_{22} - y_{12}y_{21}, y_{11}y_{23} - y_{13}y_{21}, y_{12}y_{23} - y_{13}y_{22} >$$

- *Define the Determinantal Ring*

$$R/I_l = \frac{k[y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23}]}{< y_{11}y_{22} - y_{12}y_{21}, y_{11}y_{23} - y_{13}y_{21}, y_{12}y_{23} - y_{13}y_{22} >}$$

Definition 3.3. ***(F-threshold of an Ideal)*** *Let S be a commutative ring with prime characteristic $p > 0$, and let $\mathfrak{a}$ and $\mathfrak{J}$ be ideals in S satisfying $\mathfrak{a} \subseteq \sqrt{\mathfrak{J}}$. For a fixed positive integer r, define:*

$$\nu_{\mathfrak{J}}^{\mathfrak{a}}(p^r) := \max\left\{s \in \mathbb{N} \,\middle|\, \mathfrak{a}^s \not\subseteq \mathfrak{J}^{[p^r]}\right\},$$

where $\mathfrak{J}^{[p^r]}$ is the ideal generated by the p^r-th powers of elements in $\mathfrak{J}$. The F-threshold of $\mathfrak{a}$ relative to $\mathfrak{J}$ is then given by the limit:

$$c_{\mathfrak{J}}(\mathfrak{a}) := \lim_{r\to\infty} \frac{\nu_{\mathfrak{J}}^{\mathfrak{a}}(p^r)}{p^r},$$

provided this limit exists.

Example 3.4. *Let $S = k[x, y]$, where k denote field of characteristic $p > 0$. Define the ideals: $\mathfrak{a} = (x^2, y^2)$, $\quad J = (x, y)$, then $c^J(\mathfrak{a})$ is 1.*

Definition 3.4. *The F-pure threshold of polynomail g at $\mathfrak{m}$ is then defined as*

$$\operatorname{fpt}(g) := \sup\left\{\frac{N}{p^e} \,\middle|\, g^N \notin \mathfrak{m}^{[p^e]}\right\} = \inf\left\{\frac{N}{p^e} \,\middle|\, g^N \in \mathfrak{m}^{[p^e]}\right\}.$$

where $\mathfrak{m}^{[p^e]}$ denotes the ideal generated by the p^e-th powers of the generators of $\mathfrak{m}$, i.e., $\mathfrak{m}^{[p^e]} = \langle x_1^{p^e}, \ldots, x_n^{p^e}\rangle$.

Example 3.5. *Let $g = x^2y^3 \in S = k[x, y]$, then $\operatorname{fpt}(g)$ is $1/3$.*

Definition 3.5. ***(a-invariant)*** *The a-invariant is a numerical invariant associated with a standard graded Cohen–Macaulay ring. It is defined as the top degree in which the local cohomology module $H^d_{\mathfrak{m}}(R)$, supported at the homogeneous maximal ideal $\mathfrak{m}$, is nonzero, where $d = \dim(R)$. Intuitively, it measures how far the ring is from being "positively graded" in a homological sense. Algebraically, for a standard graded Cohen–Macaulay ring R, the a-invariant is given by:*

$$a(R) = -\min\{i \mid [\omega_R]_i \neq 0\},$$

where ω_R denotes the canonical module of R, and $[\omega_R]_i$ is its degree i component. For determinantal rings generated by maximal minors, it has been observed that the F-threshold of the homogeneous maximal ideal $\mathfrak{m}$ often equals the negative of the a-invariant:

$$c^{\mathfrak{m}}(\mathfrak{m}) = -a(R).$$

Definition 3.6. ***(F-pure)*** *Let $f : R \mapsto S$ is ring homomorphism,f is said to pure if, for any R-module N ,then the map $f_M : R \otimes_R M \to S \otimes_R M$ is injective.*

Example 3.6.

$$f : \mathbb{Z} \to \mathbb{Q} \text{ is a pure ring homomorphism}$$

Definition 3.7. ***(F-finite)*** *Let R be a ring with positive characteristic is F-finite if frobenius map $F : R \mapsto R$ is a finite morphism.*

Proposition 4. *[6] Assume k is an F-finite field with prime characteristic $p > 0$. Let $Y = (y_{ij})$ be a matrix of indeterminates over k, with size $p \times q$, where $p \leq q$. For a positive integer l such that $1 \leq l \leq m$, define $I_l(Y)$ as the ideal generated by all $l \times l$ minors of Y.*

We construct the determinantal ring as:

$$R = \frac{k[Y]}{I_t(Y)},$$

and let $\mathfrak{m}$ be the maximal homogeneous ideal of R. Then, the F-pure threshold of $\mathfrak{m}$ in R is given by the formula:

$$\operatorname{fpt}(\mathfrak{m}) = p(l-1).$$

Reamark 3.1. *a-invariant of ring R is $-n(l-1)$. If $p = q$, then $\operatorname{fpt}(\mathfrak{m}) = -a(R)$.*

Proposition 5. *[5] Let R be a standard graded algebra over a field K, where the characteristic of K is a prime number $p > 0$. Assume that R is both F-finite and F-pure. The F-pure threshold of R, denoted by $\operatorname{fpt}(R)$, satisfies the following conditions for all $i \in \mathbb{N}$. Let $d = \dim(R)$, and let $a_i(R)$ be the i-th a-invariant of the ring R:*

- *For every index i, the inequality holds:*

$$\operatorname{fpt}(R) \leq -a_i(R).$$

- *If $a_i(R)$ is finite (i.e., $a_i(R) \neq -\infty$), then:*

$$-a_i(R) \leq c^m(R),$$

where $c^m(R)$ is a constant that depends on the ring R.

Proposition 6. *[3] Let R be a Noetherian ring with characteristic $p > 0$, and let $\mathfrak{m}$ be a maximal ideal of R. Suppose we have ideals $\mathfrak{a}, \mathfrak{b}, J \subseteq \mathfrak{m}$ such that $\mathfrak{a}, \mathfrak{b} \subseteq \sqrt{J}$. Now, under the assumption that $\mathfrak{a}, \mathfrak{b}, J \subseteq \mathfrak{m}$ are nonzero ideals and $\mathfrak{a}, \mathfrak{b} \subseteq \sqrt{J}$, the following properties hold:*

1. *If* $\mathfrak{a} \subseteq \mathfrak{b}$*, then we have:*

$$c_J(\mathfrak{a}) \leq c_J(\mathfrak{b}).$$

2. *For any positive integer* s*, the F-threshold scales inversely with the exponent:*

$$c_J(\mathfrak{a}^a) = \frac{1}{s} \cdot c_J(\mathfrak{a}).$$

Using the above results, we get an important inequality in the following theorem:

Theorem 3.1. *[2] Let* R *be a standard graded* K*-algebra and given standard* k*-algebra is* F-finite *and* F-pure*. Then the following chain of inequalities holds:*

$$\mathrm{fpt}(\mathfrak{m}) \leq -a(R) \leq c^{\mathfrak{m}}(R) \leq \dim(R).$$

Proof. Using above Proposition, we get inequality in this theorem. □

Chapter 4

F-threshold of Rings of Characteristic p

The F-threshold operate as a numerical invariant that measure the severity of singularities for positive characteristic. In this chapter, we discuss the some notion like δ-representation of monomail and k-special diagonal. To compute F-threshold of determinantal rings, focusing specifically on 3×3 and 4×4 matrices.

Definition 4.1. *Let δ and δ' be two diagonals, and let $f, f' \in K[y_{11}, \ldots, y_{nn}]$ be monomials. We say that*

$$f' = y_{11}^{\beta_{11}} \cdots y_{nn}^{\beta_{nn}}$$

is derived from

$$f = y_{11}^{\gamma_{11}} \cdots y_{nn}^{\gamma_{nn}}$$

by exchanging δ with δ', provided that for every $i, j \in \{1, \ldots, n\}$,

$$\beta_{ij} = \gamma_{ij} - 1_{\delta}(i,j) + 1_{\delta'}(i,j),$$

where $1_{\delta}(i,j)$ is the characteristic function, taking the value 1 if $(i,j) \in \delta$, and 0 otherwise.

Example 4.1. *Take $n = 3$ and $f \in K[y_{11}, \ldots, y_{33}]$,*
now we cosider $f = y_{11}^3 y_{22}^2 y_{33}^1$. and corresponding g is diagonal $\delta = \{(1,1),(2,2),(3,3)\}$
and $\delta' = \{(1,2),(2,2),(3,1)\}$
then by definition, we get
$f' = y_{11}^1 y_{22}^2 y_{12}^1 y_{31}^1$.

Definition 4.2. ***(Exponent matrix)*** *let* $f = {y_{11}}^{\gamma_{11}} \cdots {y_{nm}}^{\gamma_{nm}}$ *be monomail then exponent matrix denoted by M(g) and defined as:*

$$M(f) := \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nm} \end{pmatrix}$$

Definition 4.3. ***(Diagonal)*** *If* $\delta \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$*, then we call* δ *a diagonal if its elements correspond to the positions of the variables in the monomials that appear in* $\det(Y)$*. The complete collection of such diagonals is denoted by D.*

Example 4.2. *let* $M(f) := \begin{pmatrix} y_{11}^{\gamma_{11}} & y_{12}^{\gamma_{12}} \\ y_{21}^{\gamma_{21}} & y_{22}^{\gamma_{22}} \end{pmatrix}$ *then,*

$detY = y_{11}^{\beta_{11}} \cdot y_{22}^{\beta_{22}} - y_{12}^{\beta_{12}} \cdot y_{21}^{\beta_{21}}$

Here $\delta_1 = \{(1,1), (2,2)\}$

Definition 4.4. ***(Minimal intersection)*** *Consider an* $n \times n$ *square matrix* M*, and let* $\delta \in D$ *represent a diagonal. A subset* $T \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$ *is said to have minimal intersection with* δ *if, for every other diagonal* $\delta' \in D$*, the following inequality is satisfied:*

$$|T \cap \delta| \leq |T \cap \delta'|.$$

Example 4.3. *every row and cloumn of matrix have minimal intersection with every diagonal.*

Definition 4.5. ***(K-special diagonal)*** *Let* k *be any positive integer and let* $\delta \in D$ *denote a diagonal of an* $n \times n$ *matrix. Consider the monomial*

$$f = y_{11}^{\gamma_{11}} y_{22}^{\gamma_{22}} \cdots y_{nn}^{\gamma_{nn}}.$$

We define the diagonal δ *to be k-special with respect to f if, for each entry* (i, j) *in* δ*, there exists a subset*

$$T \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$$

such that:

- $(i, j) \in T$*,*
- T *has minimal intersection with* δ*, and*

- *the inequality*

$$\sum_{(r,q)\in T} \gamma_{rq} \geq (|T|-1)(k-1)+1$$

holds.

Example 4.4. *Let $k=2$, consider a monomial in 3×3 matrix,*

$$f = y_{11}^3 y_{22}^2 y_{33}^1.$$

and take diagonal

$$\delta = \{(1,1),(2,2),(3,3)\}$$

Proposition 7. *[2] Let $f \in K[y_{11},\ldots,y_{nn}]$ be a monomial appearing in the detY, and let $\delta \in D$ be any diagonal. A δ-representation of the monomail is given as:*

$$f \equiv f_1 + \cdots + f_h \pmod{g},$$

where $f_1,\ldots,f_h \in K[y_{11},\ldots,y_{nn}]$ are monomials such that, for every $i=1,\ldots,h$:

- $\deg f_i = \deg f$.

- $f_i = y_{11}^{\gamma_{11}} \ldots y_{nn}^{\gamma_{nn}}$ *is obtained by swapping with any another diagonal.*

- *It is hold*

$$\sum_{(r,q)\in\delta} \gamma_{rq} = 0.$$

Example 4.5. *Take $n=3$,and cosider $f = y_{11}^7 y_{12}^4 y_{13}^3 y_{21}^5 y_{22}^2 y_{23}^1 y_{31}^3 y_{32}^2 y_{33}^2$,$\delta = \{(1,1),(2,2),(3,3)\}$.*

$f = y_{11}^6 y_{22}^1 y_{33}^1 (y_{11}y_{22}y_{33}) y_{12}^4 y_{13}^3 y_{21}^5 y_{23}^3 y_{32}^2 y_{31}^3$.

$f = y_{11}^6 y_{22}^1 y_{33}^1 (y_{11}y_{32}y_{23} + y_{12}y_{21}y_{33} + y_{12}y_{31}y_{23} + \ldots) y_{12}^4 y_{13}^3 y_{21}^5 y_{23}^3 y_{32}^2 y_{31}^3$.

The exponent matrices for first term transform as:

$$\begin{bmatrix} 7 & 4 & 3 \\ 5 & 1 & 2 \\ 3 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 7 & 4 & 3 \\ 5 & 0 & 3 \\ 3 & 4 & 0 \end{bmatrix}.$$

similarly, The exponent matrices for second term transform as:

$$\begin{bmatrix} 6 & 5 & 3 \\ 6 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 6 & 5 & 3 \\ 6 & 0 & 2 \\ 3 & 3 & 1 \end{bmatrix}.$$

and so on.

Lemma 3. *[2] Consider the monomials $f, f' \in K[y_{11}, \ldots, y_{nn}]$. Let the exponent matrices of f and f' be $M(f) = (\gamma_{ij})$ and $M(f') = (\beta_{ij})$, respectively. Let T be a set that has minimal intersection with δ, and $\delta \in D$ be a diagonal. Assume that f' is produced from f by swapping the diagonal δ with another diagonal δ'.*

such that, we obtain the following inequality:

$$\sum_{(r,q)\in T} \gamma_{rq} \leq \sum_{(r,q)\in T} \beta_{rq}$$

If T have minimal intersection with diagonal δ', then equality holds.

Proof. we know that For every i, j,

$$\beta_{ij} = \gamma_{ij} - \mathbf{1}_{\delta}(i,j) + \mathbf{1}_{\delta'}(i,j),$$

where $\mathbf{1}_{\delta}(i,j)$ and $\mathbf{1}_{\delta'}(i,j)$ denote indicator functions, taking the value 1 if $(i,j) \in \delta$ or δ', respectively, and 0 otherwise.

Now, Take sum both sides over the set T, we obtain:

$$\sum_{(r,q)\in T} \beta_{rq} = \sum_{(r,q)\in T} \gamma_{rq} - \sum_{(r,q)\in T} \mathbf{1}_{\delta}(r,q) + \sum_{(r,q)\in T} \mathbf{1}_{\delta'}(r,q),$$

which simplifies to:

$$\sum_{(r,q)\in T} \beta_{rq} = \left(\sum_{(r,q)\in T} \gamma_{rq} \right) - |T \cap \delta| + |T \cap \delta'|.$$

Since $|T \cap \delta| \leq |T \cap \delta'|$ by assumption of minimal intersection with δ, we conclude:

$$\sum_{(r,q)\in T} \beta_{rq} \geq \sum_{(r,q)\in T} \gamma_{rq},$$

which completes the proof.

Example 4.6. *Consider the polynomial ring $K[y_{11}, y_{12}, y_{21}, y_{22}]$ over the field $K = \mathbb{Q}$. Define the monomials*

$$f = x_{11}x_{22}, \quad \textit{and} \quad f' = x_{12}x_{21}.$$

Their exponent matrices are given by:

$$M(f) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M(f') = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let the diagonal be $\delta = \{(1,1),(2,2)\}$ *and the another diagonal be* $\delta' = \{(1,2),(2,1)\}$. *Suppose* $T = \{(1,1),(2,1)\}$. *Then we can easly see that:*

$$T \cap \delta = \{(1,1)\}, \quad T \cap \delta' = \{(2,1)\},$$

each having cardinality 1.

Now compute:

$$\sum_{(r,q)\in T} \gamma_{rq} = \gamma_{11} + \gamma_{21} = 1 + 0 = 1, \qquad \sum_{(r,q)\in T} \beta_{rq} = \beta_{11} + \beta_{21} = 0 + 1 = 1.$$

then we get,

$$\sum_{(r,q)\in T} \gamma_{rq} = \sum_{(r,q)\in T} \beta_{rq},$$

and equality holds by lemma 3.

□

Proposition 8. *[2] Consider the monomial*

$$f = y_{11}^{\gamma_{11}} y_{22}^{\gamma_{22}} \cdots y_{nn}^{\gamma_{nn}} \in K[y_{11}, \ldots, y_{nn}].$$

the monomial f *belong in the ideal* $(\mathfrak{m}^{[k]}, g)$ *if there exists a diagonal* $\delta \in D$ *that is* k*-special for* f. *then*

$$f \in (\mathfrak{m}^{[k]}, g).$$

Proof. Suppose $M(f) = (\gamma_{ij})$ represent the exponent matrix of a monomial f. Suppose δ-representation modulo g, given by

$$f \equiv f_1 + \cdots + f_h \pmod{g}.$$

take any monomial f' that appears in this representation, and denote its exponent matrix by $M(f') = (\beta_{ij})$.

By definition of a δ-representation, there exists at least one position $(r,q) \in \delta$ such that $\beta_{rq} = 0$. Now, here given the diagonal δ is k-special for f. This means there is a subset $T \subseteq \{1,\ldots,n\} \times \{1,\ldots,n\}$ containing (r,q), which is minimal intersect with δ, and satisfies the inequality:

$$\sum_{(i,j)\in T} \gamma_{ij} \geq (|T| - 1)(k-1) + 1.$$

Applying Lemma 3, we get:

$$\sum_{(i,j)\in T} \beta_{ij} \geq \sum_{(i,j)\in T} \gamma ij \geq (|T| - 1)(k-1) + 1.$$

Since $\beta_{rq} = 0$, at most $|T| - 1$ of the terms β_{ij} in the sum are nonzero. Therefore, at least one of these terms must be at least k, which implies $f' \in \mathfrak{m}^{[k]}$.

Because this argument applies to every monomial f' in the δ-representation of f, we conclude that

$$f \in (\mathfrak{m}^{[k]}, g).$$

□

Reamark 4.1. (1) *Every row and column intersects any diagonal at exactly one position, which ensures that their overlap with each diagonal has minimal intersection.* (2) *Assume T is an $(n-1)$-minor corresponding to a fixed diagonal δ, and that T intersects δ in exactly $n-2$ positions. These minors are said to have minimal overlap with δ, as the definition of minimal intersection, we can say $|T \cap \delta'| \geq n-2$ for all $\delta' \in D$.*

F-threshold of determinantal ring for 3×3 matrices

Theorem 4.1. *Let $f = y_{11}^{\gamma_{11}} \cdots y_{33}^{\gamma_{33}} \in K[y_{11}, \ldots, y_{33}]$ be a monomial of total degree $6k-5$. Then:*

$$f \in (\mathfrak{m}^{[k]}, g),$$

where $\mathfrak{m}^{[k]}$ denotes the Frobenius power of the maximal ideal.

Proof. By the notion of a δ-representation, the monomial f can be expressed modulo g as:

$$f \equiv f_1 + \cdots + f_h \pmod{g},$$

where each term f_i in this representation satisfies $\deg f_i = \deg f = 6k-5$. If all such monomials f_i are contained in the ideal $(\mathfrak{m}^{[k]}, g)$, then f must also lie in the same ideal.

We may now focus on the scenario where at least one entry $\gamma_{ij} = 0$ in the exponent matrix $M(f)$. Given that the total degree is $6k-5$, there must be at least one row and one column in $M(f)$ whose entries sum to at least $2k-1$. For instance, assume:

$$\gamma_{11} + \gamma_{12} + \gamma_{13} \geq 2k-1 \quad \text{and} \quad \gamma_{11} + \gamma_{21} + \gamma_{31} \geq 2k-1.$$

All entries in the first row and first column to be nonzero, so we can assume $\gamma_{33} = 0$ without loss of generality.

Now consider the second column. If:

$$\gamma_{12} + \gamma_{22} + \gamma_{32} \geq 2k - 1,$$

then the diagonal $\delta = \{(3,1),(2,2),(1,3)\}$ is k-special for f, since each of its elements lies in a row or column summing to at least $2k-1$. Hence, by the definition of a k-special diagonal, $f \in (\mathfrak{m}^{[k]}, g)$.

Alternatively, if:

$$\gamma_{12} + \gamma_{22} + \gamma_{32} < 2k - 1,$$

then combining these with the earlier assumptions gives:

$$\gamma_{12} + \gamma_{22} + \gamma_{32} + \gamma_{31} \leq 3k - 3,$$

and since $\gamma_{33} = 0$, we conclude:

$$\gamma_{11} + \gamma_{13} + \gamma_{21} + \gamma_{23} = 6k - 5 - (\gamma_{12} + \gamma_{22} + \gamma_{32} + \gamma_{31}) \geq 3k - 2.$$

Now examine the diagonal $\delta = \{(1,2),(2,3),(3,1)\}$. This diagonal is k-special for each f_i for the following reasons:

- The first row includes the position $(1,2)$, δ minimal intersection, and its total sum is at least $2k-1$.
- Let $T = \{(1,1),(1,3),(2,1),(2,3)\}$, which satisfies $|T \cap \delta| = 1$. By Remark 4.1, this means T has minimal intersects with δ. We have:

$$\sum_{(r,q)\in T} \gamma_{rq} = \gamma_{11} + \gamma_{13} + \gamma_{21} + \gamma_{23} \geq 3k - 2 = (|T| - 1)(k-1) + 1.$$

- The first column contains the position $(3,1)$, which lies in δ and ensures minimal overlap with it. The total sum of the entries in this column is at least $2k-1$. Hence, by Proposition 8,

$$f \in (\mathfrak{m}^{[k]}, g).$$

Therefore, we conclude:

$$c_{\mathfrak{m}}^{(m)} = \lim_{e\to\infty} \frac{\nu_{p^e}(m)}{p^e} = \frac{6k-6}{k} = 6 = n(n-1), \quad \text{where } p^e = k.$$

□

F-threshold of determinantal ring for 4×4 matrices

Lemma 4. *[2] Let $M = (\gamma_{ij})$ denote the exponent matrix corresponding to a monomial $f \in K[y_{11}, \ldots, y_{44}]$ and There exists at least one row and three distinct columns, Or one column and three different rows, Or two rows along with two columns, such that in each selected row or column, the total sum of the respective entries is greater than or equal to $3k-2$. Then it follows that:*

$$f \in (\mathfrak{m}^{[k]}, g).$$

Proof. Assume that the total of the elements in at least two rows and two columns is at least $3k-2$. WLOG, we can take the first two rows and the first two columns are those whose sum of elements is at least $3k-2$. Next, the diagonal

$$\delta = \{(1,4),\ (2,3),\ (3,2),\ (4,1)\}$$

is k-special because either the first two rows or the first two columns include every element of δ. Consequently,

$$f \in (m^{[k]}, g)$$

by proposition 8. □

Lemma 5. *[2] Suppose that $f \in K[y_{11}, \ldots, y_{44}]$ is a monomial of degree $12k-11$, and that its exponent matrix is $M(f) = (\gamma_{ij})$. Let there are exactly any two rows and one column (or one row and two columns) of M with a sum of greater than equal to $3k-2$, and that atleast one of the γ_{ij} is zero. then, $f \in (m^{[k]}, g)$.*

Proof. by lemma, we assume:

(a) $\gamma_{11} + \cdots + \gamma_{14} \geq 3k-2$, $\gamma_{21} + \cdots + \gamma_{24} \geq 3k-2$, and $\gamma_{11} + \cdots + \gamma_{41} \geq 3k-2$;

(b) $\gamma_{31} + \cdots + \gamma_{34} \leq 3k-3$, $\gamma_{12} + \cdots + \gamma_{42} \leq 3k-3$, and $\gamma_{13} + \cdots + \gamma_{43} \leq 3k-3$;

(c) $\gamma_{44} = 0$.

Now, choose the diagonal $\delta = \{(1,3),(2,2),(3,4),(4,1)\}$. We analyze the properties of this diagonal with respect to the exponent matrix $M(f) = (\gamma_{ij})$:

1. From condition (a), the first row, which is contain $(1,3)$ and have minimal intersection with δ whose sum of elements is atleast $3k-2$(by Remark 4.1).

2. Similarly, the second row includes $(2,2)$ and satisfies the same minimal intersection condition with δ, whose sum of element is atleast $3k-2$.

3. Take $T = \{(1,1),(1,4),(2,1),(2,4),(3,1),(3,4)\}$, which includes $(3,4) \in \delta$. The cardinality of $S \cap \delta$ is 1. Every diagonal is intersected by this set atleast once, because every diagonal touch the first and last columns, and cannot contain both $(4,1)$ and $(4,4)$. then, by Remark 4.1 we can say T has minimal intersection with δ.

   From condition (b), we know:

$$\gamma_{12} + \cdots + \gamma_{42} \leq 3k-3 \quad \text{and} \quad \gamma_{13} + \cdots + \gamma_{43} \leq 3k-3.$$

   Then, the total cardinality of T is:

$$\sum_{(r,q)\in T} \gamma_{rq} = 12k - 11 - (\gamma_{12} + \cdots + \gamma_{42}) - (\gamma_{13} + \cdots + \gamma_{43}) - \gamma_{41}.$$

   Since $\gamma_{41} \leq k-1$, we get:

$$\sum_{(r,q)\in T} \gamma_{rq} \geq 12k-11-(3k-3)-(3k-3)-(k-1) = 5k-4 = (|T|-1)(k-1)+1.$$

4. Finally, From condition (a), the first column, which is contain $(4,1)$ and minimal intersect with δ whose sum of elements is atleast $3k-2$(by Remark 4.1)

then, the diagonal δ is k-special for f, and by Proposition 8, it follows that

$$f \in (m^{[k]}, g).$$

□

Lemma 6. *[2] Let $f \in K[y_{11},\ldots,y_{44}]$ be a monomial of total degree $12k-11$, and let $M(f) = (\gamma_{ij})$ denote its exponent matrix. Take, there is exactly one row and one column of $M(f)$ whose entries sum to greater than equal to $3k-2$, and that atleast one entry γ_{ij} is equal to zero. then, it follows that*

$$f \in (m^{[k]}, g).$$

Proof. Let the exponent matrix $M(f) = (\gamma_{ij})$ of the monomial $f \in K[y_{11},\ldots,y_{44}]$ satisfies the following:

(P1) $\gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{14} \geq 3(k-1)+1$;

(P2) $\gamma_{i1} + \gamma_{i2} + \gamma_{i3} + \gamma_{i4} \leq 3(k-1)$ for $i = 2, 3, 4$;

(Q1) $\gamma_{11} + \gamma_{21} + \gamma_{31} + \gamma_{41} \geq 3(k-1) + 1$;

(Q2) $\gamma_{1i} + \gamma_{2i} + \gamma_{3i} + \gamma_{4i} \leq 3(k-1)$ for $i = 2, 3, 4$;

(w) $\gamma_{44} = 0$.

Now, Take the diagonal $\delta = \{(1,4),(2,1),(3,3),(4,2)\}$. If this diagonal is k-special for f, then by Proposition 8, we conclude that

$$f \in (m^{[k]}, g).$$

Assume that δ is not k-special for f. Let $f \equiv f_1 + \cdots + f_h \mod f$ be a δ-representation of f, and soppose that

$$f' = x_{11}^{\beta_{11}} \cdots x_{44}^{\beta_{44}}$$

is a monomial in the δ-representation of f. By the definition of a δ-representation, there are some $(i,j) \in \delta$ such that $\beta_{ij} = 0$. We analyze the four possiblity for the zero entry and our target to prove that $f' \in (m^{[k]}, g)$ in each condition. This will imply that $f \in (m^{[k]}, g)$.

(1) Suppose $\beta_{14} = 0$. According to Remark 4.1, we can say definition of minimail intersection and lemma 3 we have;

$$\beta_{11} + \beta_{12} + \beta_{13} + \beta_{14} = \gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{14} \geq 3(k-1) + 1.$$

Since $\beta_{14} = 0$, it follows that:

$$\beta_{11} + \beta_{12} + \beta_{13} \geq 3(k-1) + 1.$$

Therefore, atleast one of them $\beta_{11}, \beta_{12}, \beta_{13}$ must be atleast k, which implies $f' \in (m^{[k]}, g)$.

(2) Let's assume that $\beta_{21} = 0$. Using the same argument as in the previous case, we obtain the following sum:

$$\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41} = \gamma_{11} + \gamma_{21} + \gamma_{31} + \gamma_{41} \geq 3(k-1) + 1.$$

Since $\beta_{21} = 0$, we get:

$$\beta_{11} + \beta_{31} + \beta_{41} \geq 3(k-1) + 1.$$

Thus, atleast one of them $\beta_{11}, \beta_{31}, \beta_{41}$ must be atleast k, then $f' \in (m^{[k]}, g)$.

(3) Suppose $\beta_{42} = 0$. Consider

$$T = \{(1,1),(1,2),(1,3),(4,1),(4,2),(4,3)\}.$$

then cardinality $T \cap \delta$ is 1. T intersects every other diagonal, we conclude that T has minimal intersection with δ. By Lemma 3, we then have:

$$\sum_{(r,q)\in T} \beta_{rq} \geq \sum_{(r,q)\in T} \gamma_{rq} = 12k - 11 - (\gamma_{21}+\gamma_{22}+\gamma_{23}+\gamma_{24}) - (\gamma_{31}+\gamma_{32}+\gamma_{33}+\gamma_{34}) - \gamma_{14}.$$

Using the assumptions $P1$, $P2$, and $\gamma_{14} \leq k-1$, we deduce:

$$\sum_{(r,q)\in T} \beta_{rq} \geq 12k - 11 - 3(k-1) - 3(k-1) - (k-1) = 5(k-1) + 1.$$

Thus, we have:

$$\beta_{11} + \beta_{12} + \beta_{13} + \beta_{41} + \beta_{43} \geq 5(k-1) + 1.$$

Since this sum involves five terms, then one of them β_{ij} atleast k. Therefore, $f' \in (m^{[k]}, g)$.

(4) Let us assume that $\beta_{33} = 0$. The diagonal

$$\delta' = \{(1,4),(2,3),(3,1),(4,2)\}$$

should be examined. Now we check the property of k-special for each index $(i,j) \in \delta'$, we confirm that δ' is k-special for f'.

We now examine each element in δ' to ensure that the required conditions for k-special are satisfied for f'.

- If $(i,j) = (1,4)$, the first row contains $(1,4)$, and by Remark 4.1, it has minimal intersection with both d and d'. Thus, by Lemma 3 and condition $P1$, we obtain:

  $$\beta_{11} + \cdots + \beta_{14} \geq \gamma_{11} + \cdots + \gamma_{14} \geq 3(k-1) + 1,$$

  and the condition is satisfied.

- If $(i,j) = (3,1)$, the first column include $(3,1)$, and by Remark 4.1, it is hold the axiom of minimal intersection with both δ and δ'. Thus, by Lemma 3 and condition $P1$, we get:

  $$\beta_{11} + \cdots + \beta_{41} \geq \gamma_{11} + \cdots + \alpha_{41} \geq 3(k-1) + 1,$$

  and our condition hold:

- If $(i,j) = (4,2)$, Take $T := \{(1,1),(1,2),(1,3),(4,1),(4,2),(4,3)\}$. From conditions $P1$, $Q2$, and Z, we deduce:

$$\gamma_{11} + \cdots + \gamma_{14} + \gamma_{41} + \cdots + \gamma_{43} \geq 6(k-1) + 1,$$

and since $\gamma_{14} \leq k-1$, we get:

$$\sum_{(r,q)\in T} \gamma_{rq} = \gamma_{11} + \cdots + \gamma_{13} + \gamma_{41} + \cdots + \gamma_{43} \geq 5(k-1) + 1.$$

Next, since T intersects every diagonal min one time then cardinality of T intersection δ. By Lemma 3 , we then obtain:

$$\sum_{(r,q)\in T} \beta_{rq} \geq \sum_{(r,q)\in T} \gamma_{rq} \geq 5(k-1) + 1,$$

and since by definition of minimal intersection our condition hold.

- If $(i,j) = (2,3)$, consider $T' := \{(1,1),(2,1),(4,1),(1,3),(2,3),(4,3)\}$. Since by same process as above, Lemma 3 and conditions $Q1$, $Q2$, we have:

$$\beta_{12}+\cdots+\beta_{42} = \gamma_{12}+\cdots+\gamma_{42} \leq 3(k-1), \quad \beta_{14}+\cdots+\beta_{44} = \gamma_{14}+\cdots+\gamma_{44} \leq 3(k-1).$$

Then, since $\beta_{33} = 0$ and $\beta_{31} \leq k-1$, we obtain:

$$\sum_{(r,q)\in T} \beta_{rq} = 12(k-1)+1-(\beta_{12}+\cdots+\beta_{42})-(\beta_{14}+\cdots+\beta_{44})-\beta_{31} \geq 5(k-1)+1.$$

Since T' has minimal intersection along δ and contains $(2,3)$, then condition is hold.

Thus, δ is k-special for f, and by Proposition 8, we conclude that $f \in (m^{[k]}, g)$. $\square$

Theorem 4.2. *Let $f \in K[y_{11},\ldots,y_{44}]$ be a monomial of degree $12k-11$. Then $f \in (m^{[k]}, g)$.*

Proof. Let $\delta \in D$ be a diagonal, and Take δ-representation of f in the form $f \equiv f_1 + \cdots + f_h \mod g$. Let f' be any monomial in this δ-representation of f. By the definition of δ-representation, we can say atleast one entry must be zero. Since $\deg(f') = 12k-11$, there must exist atleast one row and one column of $M(f')$ whose sum of entries is atleast $3k-2$. Based on the size of matrix, we consider the following cases:

case(1) If there exist three rows along with one column or three columns together with one row such that the sum of the entries in each of these selected rows and column is at least $3k-2$, then by Lemma 4, it follows that

$$f' \in (\mathfrak{m}^{[k]}, g).$$

case(2) If there are at least two rows and columns of $M(f')$ where the sum is at least $3k-2$, then $f' \in (m^{[k]}, g)$ by Lemma 4.

case(3) If there are two rows and one column or one row and two columns in the matrix $M(f')$ such that the total of the entries in each selected row and column is at least $3k-2$, then by Lemma 5, we conclude that

$$f' \in (\mathfrak{m}^{[k]}, g).$$

case(4) If there is one row and one column of $M(f')$ where the sum is at least $3k-2$, then $f' \in (m^{[k]}, f)$ by Lemma 6.

Thus, $f' \in (m^{[k]}, g)$ for all monomials f' appearing in the d-representation of f, and consequently, $f \in (m^{[k]}, g)$.
and also we get,

$$c_m{}^{(m)} = \lim_{e\to\infty} \frac{\nu_{p^e}(m)}{p^e} = \frac{12k-11}{k} = 12 = n(n-1), where\ p^e = k$$

.

□

Chapter 5

Conclusion

In this dissertation, we first recall some definitions and important results of commutative algebra and characteristic p methods. Next, we discussed the relation between the invariants of F-singularities such as F-threshold and F-fure threshold. Further we discussed about the invariant of F-sigularties such as F-threshold for a special class of rings, is called determinantal ring. Throughout the entire dissertation, we gain a better understanding of the F-threshold for rings of characteristic of p.

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